

STOCHASTIC PROOF OF UPPER BOUND FOR THE HEAT KERNEL COUPLED WITH GEOMETRIC FLOW, AND RICCI FLOW

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ABSTRACT. We give a proof of Gaussian upper bound for the heat kernel coupled with the Ricci flow. Previous proofs by Lei Ni [5] use Harnack inequality and doubling volume property, also the recent proof by Zhang and Cao [6] uses Sobolev type inequality that is conserved along Ricci flow. We will use a horizontal coupling of curve [1] Arnaudon Thalmaier, C. , in order to generalize Harnack inequality with power -for inhomogeneous heat equation - introduced by F.Y Wang. In the case of Ricci flow, we will derive on-diagonal bound of the Heat kernel along Ricci flow (and also for the usual Heat kernel on complete Manifold).

1. COUPLING AND HARNACK INEQUALITY WITH POWER

In the first part of this section, we will focus on the operator of the type $L_t := \frac{1}{2}\Delta_{g(t)}$, where $\Delta_{g(t)}$ is the Laplace operator associated to a time dependent family of metrics $g(t)$. We will suppose that all considered $g(t)$ -Brownian motion is non explosive. For example when the family of metric comes from the backward Ricci flow, this have been proved in [7]. Let $\gamma(t)$ be a $C^1(M)$ geodesic curve such that $\gamma(0) = x$ and $\gamma(1) = y$ and $X_t(u)$ be the the horizontal $L(t)$ -diffusion C^1 path space in $C^1([0, T], M)$ over X^0 , started at $u \mapsto \gamma(\frac{u}{T})$. Let $X_t(x)$ be a $g(t)$ -Brownian motion that start at x , $//_{0,t}$ the $g(t)$ parallel transport, and \mathbf{W}_t the damped parallel transport that satisfies the following Stratonovich covariant equation:

$$*d((//_{0,t})^{-1}(\mathbf{W}_{0,t})) = -\frac{1}{2}((//_{0,t})^{-1}(\text{Ric}_{g(t)} - \partial_t(g(t)))^{\#g(t)}(\mathbf{W}_{0,t}) dt$$

with

$$\mathbf{W}_{0,t} : T_x M \longrightarrow T_{X_t(x)} M, \mathbf{W}_{0,0} = \text{Id}_{T_x M}.$$

Proposition 1.1. *The process $X_t(\gamma(\frac{t}{T}))$ satisfies the following stochastic differential equation :*

$$d^{\nabla_t}(X_t(\gamma(\frac{\cdot}{T})))_t = P_{0,\frac{t}{T}}^{t,X_t(\cdot)} d^{\nabla_t} X_t^0(x) + \frac{1}{T} \mathbf{W}(X_t(\gamma(\frac{t}{T})))_t \dot{\gamma}(\frac{t}{T}) dt$$

Proof. We pass to the Stratonovich differential and obtain the following chain rule formula:

$$*d(X_t(\gamma(\frac{\cdot}{T})))_{t_0} = *d(X_t(\gamma(\frac{t_0}{T})))_{t_0} + \frac{dX_{t_0}(\frac{t}{T})}{dt} \Big|_{t=t_0} dt_0.$$

We use Theorem 3.1 in [1] to identify the last term of the right hand side:

$$\frac{dX_{t_0}(\frac{t}{T})}{dt} \Big|_{t=t_0} = \frac{1}{T} \mathbf{W}(X(\gamma(\frac{t_0}{T})))_{t_0} (\dot{\gamma}(\frac{t_0}{T})).$$

Now we come back to the Itô differential equation using the following relation:

$$d^{\nabla_t} Y_t = //_t (d \int_0^t //_s^{-1} * dY_s),$$

and we obtain

$$\begin{aligned} d^{\nabla_{t_0}} (X(\gamma(\frac{\cdot}{T})))_{t_0} &= //_{t_0} (d \int_0^{t_0} //_s^{-1} * d(X(\gamma(\frac{t_0}{T})))_s + \frac{1}{T} //_s^{-1} \mathbf{W}(X(\gamma(s)))_s (\dot{\gamma}(\frac{s}{T})) ds) \\ &= d^{\nabla_{t_0}} (X(\gamma(\frac{t_0}{T})))_{t_0} + \frac{1}{T} \mathbf{W}(X(\gamma(\frac{t_0}{T})))_{t_0} \dot{\gamma}(\frac{t_0}{T}). \end{aligned}$$

We use again Theorem 3.1 in [1] to identify

$$d^{\nabla_{t_0}} (X(\gamma(\frac{t_0}{T})))_{t_0} = P_{0, \frac{t_0}{T}}^{t_0, X_{t_0}(\cdot)} d^{\nabla_{t_0}} X_{t_0}^0(x).$$

□

Let

$$\begin{aligned} N_t &:= -\frac{1}{T} \int_0^t \langle P_{0, \frac{s}{T}}^{s, X_s(\cdot)} d^{\nabla_s} X_s^0(x), \mathbf{W}(X(\gamma(\frac{s}{T})))_s \dot{\gamma}(\frac{s}{T}) \rangle_{g(s)}, \\ R_t &:= \exp \left(N_t - \frac{1}{2} \langle N \rangle_t \right). \end{aligned}$$

In many situations the Novikov's criterion will be satisfied, so we could expect R_t to be a martingale. Define a new probability measure as :

$$\mathbb{Q} := R_T \mathbb{P}.$$

Proposition 1.2. *Suppose that Novikov's criterion is satisfied for N_t . Then under \mathbb{Q} , the process $X_t(\gamma(\frac{t}{T}))$ is a L_t -diffusion that starts at x , that finishes at $X_T(y)$.*

Proof. One could directly apply Girsanov's theorem. We prefer here to give a direct proof. Let $f \in C^2(M, \mathbb{R})$. Recall that R_t is a \mathbb{P} -martingale, and $P_{0, \frac{t}{T}}^{t, X_t(\cdot)}$ is an isometry for the metric $g(t)$. Use Itô formula to compute :

$$\begin{aligned} d(R_t f(X_t(\gamma(\frac{t}{T})))) &= \frac{1}{2} R_t \Delta_t f(X_t(\gamma(\frac{t}{T}))) dt + dM_t^{\mathbb{P}}, \end{aligned}$$

where $M_t^{\mathbb{P}}$ is a martingale for \mathbb{P} . Also

$$\begin{aligned} R_t f(X_t(\gamma(\frac{t}{T}))) - \frac{1}{2} R_t \int_0^t \Delta_s f(X_s(\gamma(\frac{s}{T}))) ds &= \\ R_t f(X_t(\gamma(\frac{t}{T}))) - \frac{1}{2} \int_0^t R_s \Delta_s f(X_s(\gamma(\frac{s}{T}))) ds + \tilde{M}_s^{\mathbb{P}}. \end{aligned}$$

Where $\tilde{M}_s^{\mathbb{P}}$ is a \mathbb{P} -martingale. Use the fact that U_t is a \mathbb{Q} -martingale if and only if $R_t U_t$ is a \mathbb{P} -martingale. So $f(X_t(\gamma(\frac{t}{T}))) - \frac{1}{2} \int_0^t \Delta_s f(X_s(\gamma(\frac{s}{T}))) ds$ is a \mathbb{Q} martingale i.e. $X_t(\gamma(\frac{t}{T}))$ is a L_t diffusion under the probability \mathbb{Q} . It is clear that it finishes at $X_T(y)$, so $X_t(\gamma(\frac{t}{T}))$ can be seen as a coupling between two L_t diffusions that started at different point up to one change of probability.

□

Let $\alpha_{i,j}(t)$ be a family of symmetric 2-tensors on M . We will consider the following heat equation coupled with a geometric flow.

$$(1.1) \quad \begin{cases} \partial_t g_{i,j} = \alpha_{i,j}(t) \\ \partial_t f(t, x) = \frac{1}{2} \Delta_t f(t, x) \\ f(0, x) = f_0(x), \end{cases}$$

Remark 1.3. Such flow, have been widely investigated in the literature. Let us mention the following situation:

- The most famous case is when $\alpha_{i,j}(t) := 0$, this is the case of constant metric and equation 1.1 is the usual heat equation in M .
- $\alpha_{i,j}(t) := -\text{Ric}_{i,j}(g(t))$, that is the Ricci flow.
- One can also consider $\alpha_{i,j}(t) := -2hH_{i,j}(g(t))$, where $H_{i,j}(g(t))$ is the second fundamental form relatively to the metric $g(t)$, and h is the mean curvature, when the family of metric comes from the mean curvature flow.

In all these cases a notion of $g(t)$ -Brownian motion, i.e. a Δ_t diffusion, parallel transport, and damped parallel transport has been given in [3, 2].

Let T_c be the maximal life time of geometric flow $g(t)_{t \in [0, T_c]}$. For all $T < T_c$, let X_t^T be a $g(T-t)$ -Brownian motion and $//_{0,t}^T$ the associated parallel transport. In this case, for a solution $f(t, \cdot)$ of (1.1), $f(T-t, X_t^T(x))$ is a local martingale for any $x \in M$. So the following representation holds for the solution :

$$f(T, x) := \mathbb{E}_x[f_0(X_T^T)].$$

We introduced a further subscript T referring to the fact that a time reversal step is involved.

Let $\mathbf{W}_{0,t}^T$ be the damped parallel transport along the $g(T-t)$ -Brownian motion. Let us recall the covariant differential equation satisfied by this damped parallel transport [3] :

$$*d((//_{0,t}^T)^{-1}(\mathbf{W}_{0,t}^T)) = -\frac{1}{2} (//_{0,t}^T)^{-1} (\text{Ric}_{g(T-t)} - \partial_t(g(T-t)))^{\#g(T-t)}(\mathbf{W}_{0,t}^T) dt$$

with

$$\mathbf{W}_{0,t}^T : T_x M \longrightarrow T_{X_t^T(x)} M, \mathbf{W}_{0,0}^T = \text{Id}_{T_x M}.$$

All the over subscript T we will mean that the family of metrics is $g(T-t)$.

Proposition 1.4. *Suppose that there exist $\overline{\alpha}, \underline{\alpha} \geq 0$ and $\overline{K}, \underline{K} \geq 0$ such that*

$$\begin{aligned} -\underline{\alpha}g(t) &\leq \alpha(t) \leq \overline{\alpha}g(t), \\ -\underline{K}g(t) &\leq \text{Ric}(t) \leq \overline{K}g(t), \end{aligned}$$

then the $g(t)$ -Brownian motion, and the $g(T-t)$ -Brownian motion does not explode before the time T_c .

Proof. This is a sufficient condition but it is far from being a necessary one, for the process to don't explode. Use the Itô formula for $d_t(x_0, X_t)$, the comparison theorem of the laplacian of the distance function, and the comparison theorem of stochastic differential equation. \square

Remark 1.5. For the backward Ricci flow, the $g(t)$ -Brownian motion does not explode [7], but the condition of the sufficient existence of the Ricci flow in complete Riemannian manifolds as given by Shi [8] theorem 1.1, that is the boundedness of the initial Riemannian tensor (for the metric $g(0)$) also gives a bound of the Ric tensor along the flow (for bounded time). So the conditions for non explosion of the $g(t)$ -Brownian motion given in the above proposition will be satisfied if the initial metric satisfy Shi condition for the complete manifolds.

Proposition 1.6. *Suppose that there exist $C \in \mathbb{R}$ such that in a matrix sens:*

$$\text{Ric}_{g(t)} - \alpha(t) \geq Cg(t).$$

Then R_t is a martingale, and for $\beta \geq 1$

$$\mathbb{E}[R_t^\beta] \leq e^{\frac{1}{2}\beta(\beta-1)\frac{d_0^2(x,y)}{T^2}\frac{1-e^{-Ct}}{C}}.$$

Moreover suppose that there exist $\tilde{C} \in \mathbb{R}$ such that in a matrix sens:

$$\text{Ric}_{g(t)} + \alpha(t) \geq \tilde{C}g(t)$$

then R_t^T is a martingale and for $\beta \geq 1$

$$\mathbb{E}[(R_t^T)^\beta] \leq e^{\frac{1}{2}\beta(\beta-1)\frac{d_T^2(x,y)}{T^2}\frac{1-e^{-\tilde{C}t}}{C}}.$$

Proof. Let $v \in T_x M$. Then we use the isometry property of the parallel transport i.e. $//_s : (T_x M, g(0)) \mapsto (T_{X_s(x)} M, g(s))$ to deduce

$$\begin{aligned} *d\langle \mathbf{W}(X.(x))_s v, \mathbf{W}(X.(x))_s v \rangle_{g(s)} &= *d\langle //_s^{-1} \mathbf{W}(X.(x))_s v, //_s^{-1} \mathbf{W}(X.(x))_s v \rangle_{g(0)} \\ &= 2\langle *d//_s^{-1} \mathbf{W}(X.(x))_s v, //_s^{-1} \mathbf{W}(X.(x))_s v \rangle_{g(0)} \\ &= 2\langle //_s *d//_s^{-1} \mathbf{W}(X.(x))_s v, \mathbf{W}(X.(x))_s v \rangle_{g(s)} \\ &= -\langle (\text{Ric}_{g(s)} - \partial_s(g(s)))^{\#g(s)}(\mathbf{W}(X.(x))_s v), \mathbf{W}(X.(x))_s v \rangle_{g(s)} ds \\ &\leq -C \| \mathbf{W}(X.(x))_s v \|^2 ds \end{aligned}$$

By Gronwall's lemma we get

$$\| \mathbf{W}(X.(x))_s v \|_{g(s)} \leq e^{-\frac{1}{2}Cs} \| v \|_{g(0)}$$

Recall that $N_t := -\frac{1}{T} \int_0^t \langle P_{0, \frac{s}{T}}^{s, X_s(\cdot)} d\nabla_s X_s^0(x), \mathbf{W}(X(\gamma(\frac{s}{T})))_s \dot{\gamma}(\frac{s}{T}) \rangle_{g(s)}$, and $P_{0, \frac{s}{T}}^{s, X_s(\cdot)}$ is a $g(s)$ isometry and $d\nabla_s X_s^0(x) = //_s e_i dw^i$ where w is a \mathbb{R}^n -Brownian motion, and $(e_i)_{i=1..n}$ is an orthonormal basis of $T_x M$. Then

$$\begin{aligned} \langle N \rangle_t &= \frac{1}{T^2} \int_0^t \| \mathbf{W}(X(\gamma(\frac{s}{T})))_s \dot{\gamma}(\frac{s}{T}) \|_{g(s)}^2 ds \\ &\leq \frac{1}{T^2} \int_0^t e^{-Cs} \| \dot{\gamma}(\frac{s}{T}) \|_{g(0)}^2 ds \\ &\leq \frac{1}{T^2} d_0^2(x, y) \int_0^t e^{-Cs} ds \end{aligned}$$

So by Nokinov's criterion, R_t is a martingale. Let $\beta \geq 1$,

$$\begin{aligned} \mathbb{E}[R_t^\beta] &= \mathbb{E}[e^{\beta N_t - \frac{\beta}{2} \langle N \rangle_t}] \\ &= \mathbb{E}[e^{\beta N_t - \frac{\beta}{2} \langle N \rangle_t} e^{\frac{\beta(\beta-1)}{2} \langle N \rangle_t}] \\ &\leq e^{\frac{1}{2} \beta(\beta-1) \frac{d_0^2(x, y)}{T^2} \frac{1-e^{-Ct}}{C}}. \end{aligned}$$

By the same computation we have

$$\begin{aligned} \langle N^T \rangle_t &= \frac{1}{T^2} \int_0^t \| \mathbf{W}^T(X(\gamma(\frac{s}{T})))_s \dot{\gamma}(\frac{s}{T}) \|_{g(T-s)}^2 ds \\ &\leq \frac{1}{T^2} \int_0^t e^{-\tilde{C}s} \| \dot{\gamma}(\frac{s}{T}) \|_{g(T)}^2 ds \\ &\leq \frac{1}{T^2} d_T^2(x, y) \int_0^t e^{-\tilde{C}s} ds. \end{aligned}$$

Thus R_t^T is a martingale. Given $\beta \geq 1$ we have similarly,

$$\mathbb{E}[(R_t^T)^\beta] \leq e^{\frac{1}{2} \beta(\beta-1) \frac{d_T^2(x, y)}{T^2} \frac{1-e^{-\tilde{C}t}}{C}}.$$

□

Remark 1.7. In the case of Ricci flow, $\partial_t g(t) = -Ric_{g(t)}$, then $\partial_t g(T-t) = Ric_{g(T-t)}$ so the process $X_t^T(x)$ does not explode (we do not need proposition 1.4, but [7]) and the condition of the above proposition is satisfied with $\tilde{C} = 0$ and

$$\mathbb{E}[(R_T^T)^\beta] \leq e^{\frac{1}{2} \beta(\beta-1) \frac{d_T^2(x, y)}{T}}.$$

We are now ready to give the Harnack inequality with power. Let f be a solution of (1.1) and let $P_{0,T}$ be the inhomogeneous heat kernel associated to (1.1), i.e.

$$P_{0,T} f_0(x) := f(T, x) = \mathbb{E}_x[f_0(X_T^T)].$$

Theorem 1.8. Suppose that the $g(T-t)$ -Brownian motion X_t^T does not explode, and that the process R_t^T is a martingale. Then : for all $\alpha > 1$ and $f_0 \in C_b(M)$ we have :

$$| P_{0,T} f_0 |^\alpha (x) \leq \mathbb{E}[(R_t^T)^{\frac{\alpha}{\alpha-1}}]^{\alpha-1} P_{0,T} | f_0 |^\alpha (y).$$

Moreover if there exists $\tilde{C} \in \mathbb{R}$ such that in a matrix sens:

$$\text{Ric}_{g(t)} + \alpha(t) \geq \tilde{C}g(t)$$

then we have:

$$| P_{0,T} f_0 |^\alpha (x) \leq e^{\frac{\alpha}{2(\alpha-1)} \frac{d_T^2(x,y)}{T^2} \frac{1-e^{-\tilde{C}T}}{\tilde{C}}} P_{0,T} | f_0 |^\alpha (y).$$

Proof. We write $\tilde{X}_t^T := X_t^T(\gamma(\frac{t}{T}))$, and use 1.2, and Holder inequality.

$$\begin{aligned} | P_{0,T} f_0 |^\alpha (x) &= | \mathbb{E}^{\mathbb{Q}}[f_0(\tilde{X}_T^T)] |^\alpha \\ &= | \mathbb{E}^{\mathbb{P}}[R_T^T f_0(\tilde{X}_T^T)] |^\alpha \\ &\leq \mathbb{E}^{\mathbb{P}}[(R_T^T)^{\frac{\alpha}{\alpha-1}}]^{\alpha-1} \mathbb{E}^{\mathbb{P}}[| f_0 |^\alpha (\tilde{X}_T^T)] \\ &= \mathbb{E}^{\mathbb{P}}[(R_T^T)^{\frac{\alpha}{\alpha-1}}]^{\alpha-1} \mathbb{E}_y^{\mathbb{P}}[| f_0 |^\alpha (X_T^T)] \\ &= \mathbb{E}^{\mathbb{P}}[(R_T^T)^{\frac{\alpha}{\alpha-1}}]^{\alpha-1} P_{0,T} | f_0 |^\alpha (y). \end{aligned}$$

The last part in the theorem is an application of proposition 1.6. \square

We will denote by μ_t the volume measure associated to the metric $g(t)$, and for A a Borelian, $V_t(A) := \int_A 1 d\mu_t$, and $B_t(x, r)$ the ball for the metric $g(t)$ of center x and radius r .

Corollary 1.9. *Suppose that the hypothesis of theorem 1.8 is satisfied, and that there exist $\tilde{C} \in \mathbb{R}$ such that $\text{Ric}_{g(t)} + \alpha(t) \geq \tilde{C}g(t)$. Let $f_0 \in L^\alpha(\mu_0)$. Moreover suppose that there exists a function $\tau : [0, T] \mapsto \mathbb{R}$ such that :*

$$\frac{1}{2} \text{trace}_{g(t)}(\alpha(t))(y) \leq \tau(t), \quad \forall (t, y) \in [0, T] \times M$$

then

$$| P_{0,T} f_0 | (x) \leq \frac{e^{\frac{\int_0^T \tau(s) ds + 1}{\alpha}}}{(V_T(B_T(x, \sqrt{\frac{2(\alpha-1)T^2}{\alpha(1-e^{-\tilde{C}T}})})))^{\frac{1}{\alpha}}} \| f_0 \|_{L^\alpha(\mu_0)}.$$

Proof. If $f_0 \in C_b(M) \cap L^\alpha(\mu_0)$ we apply theorem 1.8 and get :

$$| P_{0,T} f_0 |^\alpha (x) \leq e^{\frac{\alpha}{2(\alpha-1)} \frac{d_T^2(x,y)}{T^2} \frac{1-e^{-\tilde{C}T}}{\tilde{C}}} P_{0,T} | f_0 |^\alpha (y).$$

We integrate both sides along the ball $B_T(x, \sqrt{\frac{2(\alpha-1)T^2}{\alpha(1-e^{-\tilde{C}T}})})$, with respect to the measure μ_t , in y and obtain :

$$\begin{aligned} V_T(B_T(x, \sqrt{\frac{2(\alpha-1)T^2}{\alpha(1-e^{-\tilde{C}T}})})) | P_{0,T} f_0 |^\alpha (x) &\leq e \int_{B_T(x, \sqrt{\frac{2(\alpha-1)T^2}{\alpha(1-e^{-\tilde{C}T}})}} P_{0,T} | f_0 |^\alpha (y) d\mu_T(y) \\ &\leq e \int_M P_{0,T} | f_0 |^\alpha (y) d\mu_T(y). \end{aligned}$$

We have that $\frac{d}{dt}\mu_t(y) = \frac{1}{2} \text{trace}_{g(t)}(\alpha(t))(y)d\mu_t(y)$, and by Stokes theorem we have :

$$\begin{aligned} \frac{d}{dt} \int_M P_{0,t} |f_0|^\alpha(y) d\mu_t(y) &= \int_M P_{0,t} |f_0|^\alpha(y) \frac{d}{dt} d\mu_t(y) \\ &\leq \tau(t) \int_M P_{0,t} |f_0|^\alpha(y) d\mu_t(y). \end{aligned}$$

We deduce that :

$$\int_M P_{0,t} |f_0|^\alpha(y) d\mu_t(y) \leq e^{\int_0^t \tau(s) ds} \|f_0\|_{L^\alpha(\mu_0)}^\alpha.$$

So for $f_0 \in C_b(M) \cap L^\alpha(\mu_0)$ we have

$$|P_{0,T} f_0|(x) \leq \frac{e^{\frac{\int_0^T \tau(s) ds + 1}{\alpha}}}{\left(V_T(B_T(x, \sqrt{\frac{2(\alpha-1)T^2}{\alpha(1-e^{-CT})}}))\right)^{\frac{1}{\alpha}}} \|f_0\|_{L^\alpha(\mu_0)},$$

and we conclude by a classical density argument that the same inequality is true for $f_0 \in L^\alpha(\mu_0)$. \square

Corollary 1.10. *If the family of metric comes from the Ricci flow and if*

$$(\tau(t)) = -\frac{1}{2} \inf_{y \in M} R(t, y) < \infty, \forall t \in [0, T]$$

then we have

$$|P_{0,T} f_0|(x) \leq \frac{e^{\frac{\int_0^T \tau(s) ds + 1}{\alpha}}}{\left(V_T(B_T(x, \sqrt{\frac{2(\alpha-1)T}{\alpha}}))\right)^{\frac{1}{\alpha}}} \|f_0\|_{L^\alpha(\mu_0)}.$$

If $\inf_{x \in M} (V_T(B_T(x, \sqrt{\frac{2(\alpha-1)T}{\alpha}}))) =: C_T > 0$ then as an linear operator :

$$\|P_{0,T}\|_{L^\alpha(\mu_0) \rightarrow L^\infty(\mu_0)} \leq \frac{e^{\frac{\int_0^T \tau(s) ds + 1}{\alpha}}}{C_T^{\frac{1}{\alpha}}}$$

2. NON SYMMETRY OF THE INHOMOGENEOUS HEAT KERNEL, AND HEAT KERNEL ESTIMATE

Let $\frac{\partial}{\partial t} g(t) := \alpha(t)$ where α is a time dependent symmetric 2-tensor; and consider $L_{t,x} := -\frac{\partial}{\partial t} + \frac{1}{2} \Delta_{g(t)}$. Let $x, y \in M$ and $0 < \tau \leq \sigma \leq t$ and denote by $P(x, t, y, \tau)$ the fundamental solution of

$$(2.1) \quad \begin{cases} L_{t,x} P(x, t, y, \tau) = 0 \\ \lim_{t \searrow \tau} P(x, t, y, \tau) = \delta_x \end{cases}$$

Recall that [3]:

$$X_t^t(x) \stackrel{\mathcal{L}}{=} P(x, t, y, \tau) d\mu_0(y)$$

Let $v, u \in C^{1,2}(\mathbb{R}, M)$, that is differentiable in time, and differentiable twice in space. Consider the adjoint operator L^* of L with respect to $\langle Lu, v \rangle := \int_0^T \int_M (Lu) v d\mu_t dt$. It satisfies

$$L_{t,x}^* = \frac{1}{2} \Delta_t + \frac{\partial}{\partial t} + \frac{1}{2} \text{trace}_{g(t)}(\alpha(t)).$$

The fundamental solution $P^*(y, \tau, x, t)$ of L^* , satisfy :

$$(2.2) \quad \begin{cases} L_{\tau,y}^* P^*(y, \tau, x, t) = 0 \\ \lim_{\tau \nearrow t} P^*(y, \tau, x, t) = \delta_y \end{cases}$$

The adjoint property yields:

$$P(x, t, y, \tau) = P^*(y, \tau, x, t).$$

After a time reversal, $P^*(y, t-s, x, t)$ satisfies the following heat equation :

$$(2.3) \quad \begin{cases} \partial_s P^*(y, t-s, x, t) = \frac{1}{2} \Delta_{g(t-s),y} P^* + \frac{1}{2} \text{trace}_{g(t-s)}(\alpha(t-s))(y) P^* \\ \lim_{s \searrow 0} P^*(y, t-s, x, t) = \delta_y \end{cases}$$

Suppose that there exist functions $\tau(t)$ and $\underline{\tau}(t)$ such that :

$$\begin{aligned} \frac{1}{2} \sup_{y \in M} \text{trace}_{g(t)}(\alpha(t))(y) &\leq \tau(t) \\ \frac{1}{2} \inf_{y \in M} \text{trace}_{g(t)}(\alpha(t))(y) &\geq \underline{\tau}(t) \end{aligned}$$

By Feynman Kac formula, we conclude that :

$$P^*(y, t-s, x, t) \leq e^{\frac{1}{2} \int_0^s \tau(t-s+u) du} \overline{P}(y, s, x, t).$$

We now fix t . Let $\overline{P}(y, s, x, t)$ be the fundamental solution, defined by :

$$(2.4) \quad \begin{cases} \partial_s \overline{P}(y, s, x, t) = \frac{1}{2} \Delta_{g(t-s),y} \overline{P}(y, s, x, t) \\ \lim_{s \searrow 0} \overline{P}(y, s, x, t) = \delta_y \end{cases}$$

We have in particular:

$$X_{\frac{t}{2}}^{g(\frac{t}{2}+s)}(y) \stackrel{\mathcal{L}}{=} \overline{P}(y, \frac{t}{2}, x, t) d\mu_t(x)$$

$$(2.5) \quad \begin{cases} \partial_s f(s, x) = \frac{1}{2} \Delta_{g(t-s)} f(s, x) \\ f(0, x) = f_0(x) \end{cases}$$

Theorem 2.1. *Suppose that the $g(\frac{t}{2}+s)$ -Brownian motion does not explode before the time $\frac{t}{2}$ and that there exists $C \in \mathbb{R}$ such that $\forall s \in [0, \frac{t}{2}]$:*

$$\text{Ric}_{g(s)} - \alpha(s) \geq Cg(s).$$

Assume that $g(t-s)$ -Brownian motion does not explode before the time $\frac{t}{2}$ and that there exist $\tilde{C} \in \mathbb{R}$ such that $\forall s \in [0, \frac{t}{2}]$:

$$\text{Ric}_{g(t-s)} + \alpha(t-s) \geq \tilde{C}g(t-s).$$

Then the fundamental solution of (1.1) that we note $P(x, t, y, 0)$ satisfies :

$$P(x, t, y, 0) \leq e^{\frac{1}{2} \int_0^t \tau(s) ds} \frac{e^{-\frac{1}{2} \int_0^{\frac{t}{2}} \mathcal{I}(s) ds}}{\left(V_t(B_t(x, \sqrt{\frac{(\frac{t}{2})^2}{(1-e^{-\tilde{C}\frac{t}{2}}))}) \right)^{\frac{1}{2}} \left(V_0(B_0(y, \sqrt{\frac{(\frac{t}{2})^2}{(1-e^{-\tilde{C}\frac{t}{2}}))}) \right)^{\frac{1}{2}}}$$

Proof. By the Chapman Kolmogorov formula we have:

$$\begin{aligned} P(x, t, y, 0) &= \int_M P(x, t, z, \frac{t}{2}) P(z, \frac{t}{2}, y, 0) d\mu_{\frac{t}{2}}(z) \\ &= \int_M P(x, t, z, \frac{t}{2}) P^*(y, 0, z, \frac{t}{2}) d\mu_{\frac{t}{2}}(z) \\ &\leq \left(\int_M (P(x, t, z, \frac{t}{2}))^2 d\mu_{\frac{t}{2}}(z) \right)^{\frac{1}{2}} \left(\int_M (P^*(y, 0, z, \frac{t}{2}))^2 d\mu_{\frac{t}{2}}(z) \right)^{\frac{1}{2}}. \end{aligned}$$

Recall that $P(x, \frac{t}{2} + s, z, \frac{t}{2})$ is the fundamental solution, that starts at δ_x at time $s = 0$, of :

$$(2.6) \quad \begin{cases} \partial_s f(s, x) = \frac{1}{2} \Delta_{g(\frac{t}{2}+s)} f(s, x) \\ f(0, x) = f_0(x) \end{cases}$$

Then we have :

$$P_{0, \frac{t}{2}} f_0(x) := f_0(\frac{t}{2}, x) = \mathbb{E}[f_0(X_{\frac{t}{2}}^{t-\cdot}(x))]$$

According to the proof of corollary 1.9, we get that for $f_0 \in C_b(M) \cap L^2(\mu_{\frac{t}{2}})$:

$$| P_{0, \frac{t}{2}} f_0 | (x) \leq \frac{e^{\frac{\int_0^{\frac{t}{2}} \tau(\frac{t}{2}+s) ds + 1}{2}}}{\left(V_t(B_t(x, \sqrt{\frac{(\frac{t}{2})^2}{(1-e^{-\tilde{C}(\frac{t}{2})})})}) \right)^{\frac{1}{2}}} \| f_0 \|_{L^2(\mu_{\frac{t}{2}})}.$$

Given $x_0 \in M$ and $n \in \mathbb{N}$, we apply the above inequality to $f_0(y) := P(x, t, y, \frac{t}{2}) \wedge (nII_{B(x_0, n)}(y))$ to obtain :

$$\begin{aligned} & \int_M P(x, t, z, \frac{t}{2}) (P(x, t, z, \frac{t}{2}) \wedge nII_{B(x_0, n)}(z)) d\mu_{\frac{t}{2}}(z) \\ & \int_M (P(x, t, z, \frac{t}{2}) \wedge (nII_{B(x_0, n)}(z)))^2 d\mu_{\frac{t}{2}}(z) \\ & \leq \frac{e^{\frac{\int_0^{\frac{t}{2}} \tau(\frac{t}{2}+s) ds + 1}{2}}}{(V_t(B_t(x, \sqrt{\frac{(\frac{t}{2})^2}{(1-e^{-\tilde{C}(\frac{t}{2})})}})))^{\frac{1}{2}}} \left(\int_M (P(x, t, z, \frac{t}{2}) \wedge (nII_{B(x_0, n)}(z)))^2 d\mu_{\frac{t}{2}}(z) \right)^{\frac{1}{2}}. \end{aligned}$$

Letting n goes to infinity, we obtain that the heat kernel is twice integrable, and that:

$$\left(\int_M (P(x, t, z, \frac{t}{2}))^2 d\mu_{\frac{t}{2}}(z) \right)^{\frac{1}{2}} \leq \frac{e^{\frac{\int_0^{\frac{t}{2}} \tau(\frac{t}{2}+s) ds + 1}{2}}}{(V_t(B_t(x, \sqrt{\frac{(\frac{t}{2})^2}{(1-e^{-\tilde{C}(\frac{t}{2})})}})))^{\frac{1}{2}}}.$$

Recall that:

$$P^*(y, 0, x, \frac{t}{2}) \leq e^{\frac{1}{2} \int_0^{\frac{t}{2}} \tau(u) du} \overline{P}(y, \frac{t}{2}, x, \frac{t}{2}),$$

where $\overline{P}(y, \frac{t}{2}, x, \frac{t}{2})$ is the heat kernel at time $\frac{t}{2}$, that start at time 0 at δ_y , of the following equation:

$$(2.7) \quad \begin{cases} \partial_s f(s, x) = \frac{1}{2} \Delta_{g(\frac{t}{2}-s)} f(s, x) \\ f(0, x) = f_0(x) \end{cases}$$

We also have:

$$\overline{P}_{0, \frac{t}{2}} f_0(x) := f_0(\frac{t}{2}, x) = \mathbb{E}[f_0(X_{\frac{t}{2}}^{g(\cdot)}(x))]$$

Here the family of metric is $s \mapsto g(\frac{t}{2} - s)$, so many changes of sign are involved. However, the proof of the following is the same as the proof of corollary 1.9. We get for $f_0 \in C_b(M) \cap L^2(\mu_{\frac{t}{2}})$:

$$|\overline{P}_{0, \frac{t}{2}} f_0|(y) \leq \frac{e^{\frac{-\int_0^{\frac{t}{2}} \tau(s) ds + 1}{2}}}{(V_0(B_0(y, \sqrt{\frac{(\frac{t}{2})^2}{(1-e^{-C(\frac{t}{2})})}})))^{\frac{1}{2}}} \|f_0\|_{L^2(\mu_{\frac{t}{2}})}.$$

Similarly we can show the square integrability of the kernel and the following inequality:

$$\left(\int_M \left(\overline{P}(y, \frac{t}{2}, z, \frac{t}{2}) \right)^2 d\mu_{\frac{t}{2}}(z) \right)^{\frac{1}{2}} \leq \frac{e^{\frac{-\int_0^{\frac{t}{2}} \underline{\tau}(s) ds + 1}{2}}}{\left(V_0(B_0(y, \sqrt{\frac{(\frac{t}{2})^2}{(1-e^{-C(\frac{t}{2})})}})) \right)^{\frac{1}{2}}}.$$

We obtain :

$$\begin{aligned} & \left(\int_M (P^*(y, 0, z, \frac{t}{2}))^2 d\mu_{\frac{t}{2}}(z) \right)^{\frac{1}{2}} \\ & \leq e^{\frac{1}{2} \int_0^{\frac{t}{2}} \tau(u) du} \left(\int_M \overline{P}^2(y, \frac{t}{2}, z, \frac{t}{2}) d\mu_{\frac{t}{2}}(z) \right)^{\frac{1}{2}} \\ & \leq e^{\frac{1}{2} \int_0^{\frac{t}{2}} \tau(u) du} \frac{e^{\frac{-\int_0^{\frac{t}{2}} \underline{\tau}(s) ds + 1}{2}}}{\left(V_0(B_0(y, \sqrt{\frac{(\frac{t}{2})^2}{(1-e^{-C(\frac{t}{2})})}})) \right)^{\frac{1}{2}}}. \end{aligned}$$

□

Remark 2.2. Having a heat kernel estimate for the heat equation we have simultaneously a kernel estimate of conjugate equation.

Remark 2.3. If $g(t) = g(0)$ is constant, and $Ric_{g(0)} \geq 0$ we have $\tau(t) = \underline{\tau}(t) = 0$, $C = \tilde{C} = 0$ and we deduce Li Yau one diagonal estimate of the usual heat equation on complete manifolds:

$$P_t(x, y) \leq e^{\frac{1}{(V_0(B_0(x, \sqrt{\frac{t}{2}})))^{\frac{1}{2}}}} \frac{1}{(V_0(B_0(y, \sqrt{\frac{t}{2}})))^{\frac{1}{2}}}}$$

Remark 2.4. For the Ricci flow, in dimension 3, there is a result of Hamilton that says : if the Ricci curvature is non negative at time 0, it is still non negative for all time before the critical time. In arbitrary dimension n , if we suppose that the Ricci curvature is positive at all times, we have the following one diagonal estimate (use the above theorem 2.1 with $C = 0$ and $\tilde{C} = 0$) :

$$P(x, t, y, 0) \leq e^{\frac{e^{\frac{1}{2} \int_0^t \tau(s) ds}}{(V_t(B_t(x, \sqrt{\frac{t}{2}})))^{\frac{1}{2}}}} \frac{e^{-\frac{1}{2} \int_0^{\frac{t}{2}} \underline{\tau}(s) ds}}{(V_0(B_0(y, \sqrt{\frac{t}{2}})))^{\frac{1}{2}}}.$$

Recall that in the case of Ricci flow : $\tau(s) = -\frac{1}{2} \inf_M R(s, \cdot)$ and $\underline{\tau}(s) = -\frac{1}{2} \sup_M R(s, \cdot)$ so we have :

$$P(x, t, y, 0) \leq e^{\frac{e^{-\frac{1}{4} \int_{\frac{t}{2}}^t \inf_M R(s, \cdot) ds}}{(V_t(B_t(x, \sqrt{\frac{t}{2}})))^{\frac{1}{2}}}} \frac{e^{\frac{1}{4} \int_0^{\frac{t}{2}} (\sup_M R(s, \cdot) - \inf_M R(s, \cdot)) ds}}{(V_0(B_0(y, \sqrt{\frac{t}{2}})))^{\frac{1}{2}}}.$$

3. GRIGOR'YAN TRICKS, ONE DIAGONAL ESTIMATE TO GAUSSIAN ESTIMATE, THE RICCI FLOW CASE

In this section we use the one diagonal estimate of the previous section to derive a Gaussian type estimate of the heat kernel coupled with Ricci flow (for complete manifold with non negative Ricci curvature) . The proof involves in several steps. In particular, we use a modification of Grigor'yan tricks to control exponential integrability of the square of the heat kernel, combined to an adapted version of Hamilton entropy estimate to control the difference of the heat kernel at two points. This type of strategy, is a modification of different arguments which appears in the literature on the Ricci flow (Hamilton, Lei Ni , Cao-Zhang). Unfortunately, we have not been able to cover the case of general manifold (without assumption of non negativity of the Ricci curvature).

We start this section by the following entropy estimate.

Lemma 3.1. *Let f a positive solution of (1.1), where $\alpha_{i,j}(t) = -(\text{Ric}_{g(t)})_{i,j}$, $t \in [0, ..T_c[$ and $M_{\frac{t}{2}} := \sup_{x \in M} f(\frac{t}{2}, x)$ then for all $x, y \in M$*

$$f(t, x) \leq \sqrt{f(t, y)} \sqrt{M_{\frac{t}{2}} e^{\frac{d_t^2(x, y)}{t}}}.$$

Proof. By the homogeneity of the desired inequality under multiplication by a constant, and the homogeneity of the heat equation under the same operation, we can suppose that $f > 1$, in the proof.

Using an orthonormal frame and Weitzenbock formula, we have the following identity:

$$(-\partial_t + \frac{1}{2}\Delta_{g(t)})\left(\frac{\|\nabla f\|^2}{f}\right)(x, t) = \frac{1}{f}\left(\|Hess f - \frac{\nabla^t f \otimes \nabla f}{f}\|_{HS}^2 + (\text{Ric}_t + \dot{g})(\nabla f; \nabla f)\right)(x, t).$$

Thus, in the case of Ricci flow we get :

$$(-\partial_t + \frac{1}{2}\Delta_{g(t)})\left(\frac{\|\nabla f\|^2}{f}\right) \geq 0.$$

By a direct computation we have :

$$(-\partial_t + \frac{1}{2}\Delta_{g(t)})(f \log f)(x, t) = \frac{1}{2} \frac{\|\nabla f\|^2}{f}(x, t).$$

Let

$$N_s := h(s)\left(\frac{\|\nabla f\|^2}{f}\right)(t-s, X_s^t(x)) + (f \log f)(t-s, X_s^t(x)),$$

where $X_s^t(x)$ is a $g(t-s)$ -Brownian motion started at x . If $h(s) := \frac{t/2-s}{2}$ then by Itô formula, it is easy to see that N_s is a super-martingale. So we have :

$$\mathbb{E}[N_0] \leq \mathbb{E}[N_{\frac{t}{2}}],$$

that is :

$$\begin{aligned} \frac{t}{4} \frac{\|\nabla f\|^2}{f}(t, x) + (f \log f)(t, x) &\leq \mathbb{E}[(f \log f)(\frac{t}{2}, X_{\frac{t}{2}}^t(x))] \\ &\leq \mathbb{E}[f(\frac{t}{2}, X_{\frac{t}{2}}^t(x))] \log(M_{\frac{t}{2}}) \\ &= f(t, x) \log(M_{\frac{t}{2}}). \end{aligned}$$

Where we have used that $f > 1$ and that $f(t-s, X_s^t(x))$ is a martingale. The above computation yields:

$$\frac{\|\nabla f\|}{f}(t, x) \leq \frac{2}{\sqrt{t}} \log\left(\frac{M_{\frac{t}{2}}}{f(t, x)}\right),$$

and consequently:

$$\|\nabla \sqrt{\log\left(\frac{M_{\frac{t}{2}}}{f(x, t)}\right)}\|_t \leq \frac{1}{\sqrt{t}}.$$

After integrating this inequality along a $g(t)$ geodesic between x and y , we get :

$$\sqrt{\log\left(\frac{M_{\frac{t}{2}}}{f(y, t)}\right)} \leq \sqrt{\log\left(\frac{M_{\frac{t}{2}}}{f(x, t)}\right)} + \frac{d_t(x, y)}{\sqrt{t}},$$

that is

$$f(t, x) \leq \sqrt{f(t, y)} \sqrt{M_{\frac{t}{2}}} e^{\frac{d_t^2(x, y)}{t}}.$$

□

Now, we adapt the argument of Grigor'yan to the situation of Ricci flow (with non negative Ricci curvature). We begin by recalling Remark 2.4. The assumption of the positivity of the Ricci curvature gives $R(x, s) \geq 0$. Let $\Psi(t) := e^{\frac{1}{4} \int_0^t \sup_M R(s, \cdot) ds}$ so the estimate in 2.4 becomes :

$$P(x, t, y, 0) \leq e^{\frac{\Psi(t)}{(V_t(B_t(x, \sqrt{\frac{t}{2}})))^{\frac{1}{2}} (V_0(B_0(y, \sqrt{\frac{t}{2}})))^{\frac{1}{2}}}.$$

Lemma 3.2. *Suppose that the family of metrics $g(t)$ comes from the Ricci flow, and let B be a Borelian in M . Then:*

$$\frac{1}{V_t(B)^{\frac{1}{2}}} \leq \frac{\Psi(t)}{V_0(B)^{\frac{1}{2}}}.$$

Moreover if $\text{Ric}_{g(\cdot)} \geq 0$ then for all $x, y \in M$ and $r > 0$ we have :

$$\frac{1}{V_0(B_t(x, r))^{\frac{1}{2}}} \leq \frac{1}{V_0(B_0(x, r))^{\frac{1}{2}}}.$$

Proof. A simple computation shows that:

$$\frac{d}{dt} \mu_t = \frac{1}{2} \text{trace}_{g(t)}(\dot{g}(t)) \mu_t.$$

In the case of a Ricci flow this becomes $\frac{d}{dt}\mu_t(dx) = -\frac{1}{2}R(x,t)\mu_t(dx)$. Thus, the first inequality of the lemma follows from a direct integration. For the second point, it's clear that $\text{Ric} \geq 0$ yields that $d_t(x, y)$ is non increasing in time. So $B_0(x, r) \subset B_t(x, r)$, which clearly gives $\frac{1}{V_0(B_t(x, r))^{\frac{1}{2}}} \leq \frac{1}{V_0(B_0(x, r))^{\frac{1}{2}}}$. \square

The above lemma immediately yields the following remark.

Remark 3.3. If $\text{Ric}_{g(\cdot)} \geq 0$, and $\dot{g}(t) = -\text{Ric}_{g(t)}$ then we have:

$$P(x, t, y, 0) \leq e^{\frac{\Psi(t)^2}{(V_0(B_0(x, \sqrt{\frac{t}{2}})))^{\frac{1}{2}}(V_0(B_0(y, \sqrt{\frac{t}{2}})))^{\frac{1}{2}}}}.$$

Proposition 3.4. Let $g(t)$ be a solution of Ricci flow such that $\text{Ric}_g(t) \geq 0$, and $r > 0$, $t_0 > t \geq 0$, $A \geq 1$. Let also :

$$\xi(y, t) = \begin{cases} \frac{-(r-d_t(x, y))^2}{A(t_0-t)} & \text{if } d_t(x, y) \leq r \\ 0 & \text{if } d_t(x, y) \geq r \end{cases}$$

and $\Lambda(t) = \int_0^t \inf_{x \in M} (R(s, x)) ds$. Then for $f(t, x)$ a solution of (1.1) we have for $t_2 < t_1 < t_0$:

$$\int_M f^2(t_1, y) e^{\xi(y, t_1)} \mu_{t_1}(dy) \leq e^{-(\Lambda(t_1) - \Lambda(t_2))} \int_M f^2(t_2, y) e^{\xi(y, t_2)} \mu_{t_2}(dy).$$

Proof. By direct computation and using intensively that $\text{Ric}_{g(t)} \geq 0$. \square

Let

$$I_r(t) := \int_{M \setminus B_t(x, r)} f^2(t, y) \mu_t(dy).$$

Proposition 3.5. Under the same assumptions as in the above proposition, and if, $\rho < r$ we have:

$$I_r(t_1) \leq e^{-(\Lambda(t_1) - \Lambda(t_2))} (I_\rho(t_2) + e^{\frac{-(r-\rho)^2}{A(t_1-t_2)}} \int_M f^2(t_2, y) \mu_{t_2}(dy)).$$

Proof.

$$\begin{aligned}
I_r(t_1) &:= \int_{M \setminus B_{t_1}(x,r)} f^2(t_1, y) \mu_{t_1}(dy) \\
&\leq \int_{M \setminus B_{t_1}(x,r)} f^2(t_1, y) e^{\xi(y, t_1)} \mu_{t_1}(dy) \\
&\leq \int_M f^2(t_1, y) e^{\xi(y, t_1)} \mu_{t_1}(dy) \\
&\leq e^{-(\Lambda(t_1) - \Lambda(t_2))} \int_M f^2(t_2, y) e^{\xi(y, t_2)} \mu_{t_2}(dy) \\
&\leq e^{-(\Lambda(t_1) - \Lambda(t_2))} \left(\int_{B_{t_2}(x, \rho)} f^2(t_2, y) e^{\xi(y, t_2)} \mu_{t_2}(dy) \right. \\
&\quad \left. + \int_{M \setminus B_{t_2}(x, \rho)} f^2(t_2, y) e^{\xi(y, t_2)} \mu_{t_2}(dy) \right) \\
&\leq e^{-(\Lambda(t_1) - \Lambda(t_2))} \left(I_\rho(t_2) + \int_{B_{t_2}(x, \rho)} f^2(t_2, y) e^{\xi(y, t_2)} \mu_{t_2}(dy) \right) \\
&\leq e^{-(\Lambda(t_1) - \Lambda(t_2))} \left(I_\rho(t_2) + e^{\frac{-(r-\rho)^2}{A(t_0-t_2)}} \int_M f^2(t_1, y) \mu_{t_2}(dy) \right)
\end{aligned}$$

Then remark that the definition of $I_r(t)$ is independent of t_0 and of the corresponding ξ , so we can pass to the limit when $t_0 \searrow t_1$ in the following :

$$I_r(t_1) \leq e^{-(\Lambda(t_1) - \Lambda(t_2))} \left(I_\rho(t_2) + e^{\frac{-(r-\rho)^2}{A(t_0-t_2)}} \int_M f^2(t_1, y) \mu_{t_2}(dy) \right)$$

to obtain the desired result. \square

We apply the above proposition to the heat kernel $P(x, t, y, 0)$ of the equation (2.1) that also satisfy (1.1).

Theorem 3.6. *If $\dot{g}(t) = -\text{Ric}_{g(t)}$ and $\text{Ric}_{g(t)} \geq 0$, and the following technical assumption is satisfied for the initial manifold*

- *H1 : if M is not compact, we suppose that there exists a uniform constant $c_n > 0$ such that $\text{Vol}(B_{g(0)}(x, r)) \geq c_n r^n$ (that is a non collapsing condition)*
- *H2 : all the curvature tensor is bounded for the metric $(M, g(0))$.*

Then for all $a > 1$ there exist two positive explicit constants q_a, m_a depending only on a and on the dimension, such that we have the following heat kernel estimate :

$$P(y_0, t, x_0, 0) \leq q_a \frac{e^{\int_0^t \frac{5}{8} \sup_M R(u, \cdot) - \frac{1}{4} \inf_M R(u, \cdot) du}}{(V_0(B_0(x_0, \sqrt{t})))^{\frac{1}{2}} V_0(B_0(y_0, \sqrt{t}))^{\frac{1}{2}}} e^{-\frac{d_t(x_0, y_0)^2}{m_a t}}.$$

Proof. Let $f(t, x) := P(x, t, y, 0)$ be the heat kernel of (1.1) that is the solution of equation (2.1). Then we have by the proof of theorem 2.1:

$$\begin{aligned}
\int_M f^2(t, x) \mu_t(dx) &= \int_M P^2(x, t, y, 0) \mu_t(dx) \\
&= \int_M P^{*2}(y, 0, x, t) \mu_t(dx) \\
&\leq e^{\frac{\int_0^t \tau(u) - \underline{\tau}(u) du}{(V_0(B_0(y, \sqrt{t})))}} \cdot \\
&= e^{\frac{\int_0^t \sup_M R(u, \cdot) - \inf_M R(u, \cdot) du}{(V_0(B_0(y, \sqrt{t})))}}
\end{aligned}$$

Let $0 < \rho < r$, $A \geq 1$ and $t_2 < t_1 < t_0$ then apply proposition 3.5 to $f(t, x) := P(x, t, y, 0)$, to get :

$$\begin{aligned}
I_r(t_1) &\leq e^{-(\Lambda(t_1) - \Lambda(t_2))} (I_\rho(t_2) + e^{\frac{-(r-\rho)^2}{A(t_1-t_2)}} \int_M f^2(t_2, y) \mu_{t_2}(dy)) \\
&\leq e^{-(\Lambda(t_1) - \Lambda(t_2))} (I_\rho(t_2) + e e^{\frac{-(r-\rho)^2}{A(t_1-t_2)}} \frac{\int_0^{t_2} \sup_M R(u, \cdot) - \inf_M R(u, \cdot) du}{(V_0(B_0(y, \sqrt{t_2})))})
\end{aligned}$$

Let $a > 1$ be a constant. Following Gregorian we define : $r_k := (\frac{1}{2} + \frac{t}{k+2})r$ and $t_k := \frac{1}{a^k}$.

Thus proposition 3.5 can be applied to $r_{k+1} < r_k$ and $t_{k+1} < t_k$, yielding to the same estimate as before :

$$\begin{aligned}
I_{r_k}(t_k) &\leq e^{-(\Lambda(t_k) - \Lambda(t_{k+1}))} (I_{r_{k+1}}(t_{k+1}) + e e^{\frac{-(r_k - r_{k+1})^2}{A(t_k - t_{k+1})}} \frac{\int_0^{t_{k+1}} \sup_M R(u, \cdot) - \inf_M R(u, \cdot) du}{(V_0(B_0(y, \sqrt{t_{k+1}})))}) \\
&\leq e^{-(\Lambda(t_k) - \Lambda(t_{k+1}))} (I_{r_{k+1}}(t_{k+1}) + e e^{\frac{-(r_k - r_{k+1})^2}{A(t_k - t_{k+1})}} \frac{\int_0^{t_{k+1}} \sup_M R(u, \cdot) du - \Lambda(t_{k+1})}{(V_0(B_0(y, \sqrt{t_{k+1}})))})
\end{aligned}$$

Applying recursively this inequality, we have for all k :

(3.1)

$$\begin{aligned}
I_{r_0}(t_0) &\leq e^{-(\Lambda(t_0) - \Lambda(t_{k+1}))} I_{r_{k+1}}(t_{k+1}) + e \sum_{i=0}^k e^{-\Lambda(t_0)} e^{\frac{-(r_i - r_{i+1})^2}{A(t_i - t_{i+1})}} \frac{\int_0^{t_{i+1}} \sup_M R(u, \cdot) du}{(V_0(B_0(y, \sqrt{t_{i+1}})))} \\
&\leq e^{-\Lambda(t_0) - \Lambda(t_{k+1})} I_{r_{k+1}}(t_{k+1}) + e e^{-\Lambda(t_0)} e^{\int_0^{t_0} \sup_M R(u, \cdot) du} \sum_{i=0}^k e^{\frac{-(r_i - r_{i+1})^2}{A(t_i - t_{i+1})}} \frac{1}{(V_0(B_0(y, \sqrt{t_{i+1}})))} ,
\end{aligned}$$

where in the last inequality we use that $Ric_{g(t)} \geq 0$ so $R(x, t) \geq 0$.

We have $\lim_{k \rightarrow \infty} I_{r_k}(t_k) = 0$, by Levis asymptotic of heat kernel. This can be seen, using a probabilistic argument: use that for small t , $\mathbb{P}_x(\tau_r <$

$t) \leq Ce^{-\frac{r}{2t}}$, where $\tau_r := \inf t > 0, d_t(X_t(x), x) = r$, and

$$\begin{aligned} \int_{(M \setminus B_t(x, r))} P^2(x, t, y, 0) \mu_t(dy) &\leq \frac{cst}{t^{\frac{n}{2}}} \int_{(M \setminus B_t(x, r))} P(x, t, y, 0) \mu_t(dy) \\ &\leq cst \frac{\mathbb{P}_x(\tau_r < t)}{t^{\frac{n}{2}}} \\ &\leq cst \frac{e^{-\frac{r}{2t}}}{t^{\frac{n}{2}}} \end{aligned}$$

and the right hand side goes to 0 when t goes to 0. (we used H1 to get a global bound of the heat kernel i.e. remark 3.3).

So we can pass to the limit when k goes to infinity in equation (3.1) to get :

$$I_{r_0}(t_0) \leq ee^{-\Lambda(t_0)} e^{\int_0^{t_0} \sup_M R(u, \cdot) du} \sum_{i=0}^{\infty} e^{\frac{-(r_i - r_{i+1})^2}{A(t_i - t_{i+1})}} \frac{1}{(V_0(B_0(y, \sqrt{t_{i+1}})))}.$$

Recall that $r_i - r_{i+1} = \frac{r}{(i+3)(i+2)}$ and $t_i - t_{i+1} = \frac{t}{a^i}(1 - \frac{1}{a})$. Also By Bishop-Gromov theorem in the case $\text{Ric} \geq 0$ we have

$$\frac{V_0(B_0(y, \sqrt{t_i}))}{V_0(B_0(y, \sqrt{t_{i+1}}))} \leq a^{\frac{n}{2}} := c_a.$$

Iterating the above inequality we get :

$$\frac{V_0(B_0(y, \sqrt{t_0}))}{V_0(B_0(y, \sqrt{t_{i+1}}))} \leq (c_a)^{i+1}.$$

So we have :

$$\begin{aligned} I_{r_0}(t_0) &\leq e \frac{e^{-\Lambda(t_0)} e^{\int_0^{t_0} \sup_M R(u, \cdot) du}}{V_0(B_0(y, \sqrt{t_0}))} \sum_{i=0}^{\infty} e^{\frac{-(r_i - r_{i+1})^2}{A(t_i - t_{i+1})}} (c_a)^{i+1} \\ &\leq e \frac{e^{-\Lambda(t_0)} e^{\int_0^{t_0} \sup_M R(u, \cdot) du}}{V_0(B_0(y, \sqrt{t_0}))} \sum_{i=0}^{\infty} e^{\frac{-\left(\frac{r}{(i+3)(i+2)}\right)^2}{A\left(\frac{t}{a^i}\left(1 - \frac{1}{a}\right)\right)} + (i+1)\log(c_a)} \\ &\leq e \frac{e^{-\Lambda(t_0)} e^{\int_0^{t_0} \sup_M R(u, \cdot) du}}{V_0(B_0(y, \sqrt{t_0}))} \sum_{i=0}^{\infty} e^{\frac{-a^{i+1}r^2}{At_0(a-1)(i+3)^4} + (i+1)\log(c_a)}. \end{aligned}$$

There exists a constant $m_{(a,A)}$ such that $\frac{a^{i+1}}{A(a-1)(i+3)^4} \geq m_{(a,A)}(i+2)$, and thus we get :

$$\begin{aligned} I_{r_0}(t_0) &\leq e \frac{e^{-\Lambda(t_0)} e^{\int_0^{t_0} \sup_M R(u, \cdot) du}}{V_0(B_0(y, \sqrt{t_0}))} \sum_{i=0}^{\infty} e^{\frac{-m_{(a,A)}r^2}{t_0}(i+2) + (i+1)\log(c_a)} \\ &\leq e \frac{e^{-\Lambda(t_0)} e^{\int_0^{t_0} \sup_M R(u, \cdot) du}}{V_0(B_0(y, \sqrt{t_0}))} e^{\frac{-m_{(a,A)}r^2}{t_0}} \sum_{i=0}^{\infty} e^{-(i+1)\left(\frac{m_{(a,A)}r^2}{t_0} - \log(c_a)\right)}. \end{aligned}$$

If $\frac{m_{(a,A)}r^2}{t_0} - \log(c_a) \geq \log(2)$ then

$$I_{r_0}(t_0) \leq e^{\frac{e^{-\Lambda(t_0)}e^{\int_0^{t_0} \sup_M R(u,\cdot) du}{V_0(B_0(y, \sqrt{t_0}))} - \frac{m_{(a,A)}r^2}{t_0}},$$

If $\frac{m_{(a,A)}r^2}{t_0} - \log(c_a) < \log(2)$ then

$$\begin{aligned} I_{r_0}(t_0) &\leq \int_M P^2(x, t_0, y, 0) \mu_{t_0}(dx) \\ &= \int_M P^{*2}(y, 0, x, t_0) \mu_{t_0}(dx) \\ &\leq e^{\frac{e^{\int_0^{t_0} \sup_M R(u,\cdot) - \inf_M R(u,\cdot) du}}{(V_0(B_0(y, \sqrt{t_0})))}} \\ &\leq e^{\frac{e^{\int_0^{t_0} \sup_M R(u,\cdot) - \inf_M R(u,\cdot) du}}{(V_0(B_0(y, \sqrt{t_0})))}} e^{\log(2) + \log(c_a) - \frac{m_{(a,A)}r^2}{t_0}}. \end{aligned}$$

Take $A = 1$, we have that for all $a > 1$ there exists a constant $q_a := 2ea^{\frac{n}{2}}$ and $m_a := m_{(a,1)}$ such that :

$$(3.2) \quad I_r(t) \leq q_a \frac{e^{\int_0^t \sup_M R(u,\cdot) - \inf_M R(u,\cdot) du}}{(V_0(B_0(y, \sqrt{t_0})))} e^{-\frac{m_a r^2}{t}}$$

Let $x_0, y_0 \in M$ such that $d_t(x_0, y_0) \geq \sqrt{t}$, let $r := \frac{d_t(x_0, y_0)}{2}$, then by (3.2) (with $I_r(t)$ defined with $f(t, x) = P^2(x, t, x_0, 0)$), there exists $z_0 \in B_t(y_0, \sqrt{\frac{t}{4}}) \subset M \setminus B_t(x_0, r)$ such that :

$$\begin{aligned} V_t(B_t(y_0, \sqrt{\frac{t}{4}})) P^2(z_0, t, x_0, 0) &\leq I_r(t) \\ &\leq q_a \frac{e^{\int_0^t \sup_M R(u,\cdot) - \inf_M R(u,\cdot) du}}{(V_0(B_0(x_0, \sqrt{t})))} e^{-\frac{m_a d_t(x_0, y_0)^2}{4t}}. \end{aligned}$$

So there exists $z_0 \in B_t(y_0, \sqrt{\frac{t}{4}})$ such that:

$$P^2(z_0, t, x_0, 0) \leq q_a \frac{e^{\int_0^t \sup_M R(u,\cdot) - \inf_M R(u,\cdot) du}}{(V_0(B_0(x_0, \sqrt{t})) V_t(B_t(y_0, \sqrt{\frac{t}{4}})))} e^{-\frac{m_a d_t(x_0, y_0)^2}{4t}}$$

We denote q_a a constant that depends only of the parameter a and the dimensions, that possibly changes from line by line. By the above lemma (comparison of volume) we have :

$$P(z_0, t, x_0, 0) \leq q_a \psi(t) \frac{e^{\frac{1}{2} \int_0^t \sup_M R(u,\cdot) - \inf_M R(u,\cdot) du}}{\sqrt{(V_0(B_0(x_0, \sqrt{t})) V_0(B_0(y_0, \sqrt{\frac{t}{4}})))}} e^{-\frac{m_a d_t(x_0, y_0)^2}{8t}}$$

We conclude the proof by using lemma 3.1 (for $f(t, x) := P(x, t, x_0, 0)$) to compare the solution of the heat equation at different point. We have :

$$\begin{aligned}
P(y_0, t, x_0, 0) &\leq \sqrt{P(z_0, t, x_0, 0)} \sqrt{\sup_M P(\cdot, \frac{t}{2}, x_0, 0) e^{\frac{d_t(z_0, y_0)^2}{t}}} \\
&\leq \sqrt{P(z_0, t, x_0, 0)} \sqrt{\sup_M P(\cdot, \frac{t}{2}, x_0, 0) e^{\frac{1}{4}}} \\
&\leq q_a \sqrt{P(z_0, t, x_0, 0)} \sqrt{\frac{\Psi(\frac{t}{2})^2}{(V_0(B_0(x_0, \sqrt{\frac{t}{2}})))^{\frac{1}{2}} (t)^{\frac{n}{4}}}} \quad \text{use rmq3.3 and H1} \\
&\leq q_a \frac{\psi(t)^{\frac{3}{2}} e^{\frac{1}{4} \int_0^t \sup_M R(u, \cdot) - \inf_M R(u, \cdot) du}}{(V_0(B_0(x_0, \sqrt{\frac{t}{2}})))^{\frac{1}{2}} V_0(B_0(y_0, \sqrt{\frac{t}{4}}))^{\frac{1}{4}} t^{\frac{n}{8}}} e^{-\frac{m_a d_t(x_0, y_0)^2}{16t}} \\
&\leq q_a \frac{\psi(t)^{\frac{3}{2}} e^{\frac{1}{4} \int_0^t \sup_M R(u, \cdot) - \inf_M R(u, \cdot) du}}{(V_0(B_0(x_0, \sqrt{\frac{t}{2}})))^{\frac{1}{2}} V_0(B_0(y_0, \sqrt{\frac{t}{2}}))^{\frac{1}{2}}} e^{-\frac{m_a d_t(x_0, y_0)^2}{16t}}
\end{aligned}$$

where in the two last inequalities we use Bishop-Gromov theorem volume comparison theorem to compare volume of ball in positive Ricci curvature case to the corresponding Euclidean volume :

$$\frac{1}{r^n} \leq \frac{cst_n}{v_0(B_0(x, r))} \text{ and } \frac{1}{v_0(B_0(x, r))} \leq \frac{cst_n(\lambda)}{v_0(B_0(x, \lambda r))} \text{ for } \lambda \geq 1.$$

By the same argument we have in a more natural way :

$$P(y_0, t, x_0, 0) \leq q_a \frac{e^{\int_0^t \frac{5}{8} \sup_M R(u, \cdot) - \frac{1}{4} \inf_M R(u, \cdot) du}}{(V_0(B_0(x_0, \sqrt{t})))^{\frac{1}{2}} V_0(B_0(y_0, \sqrt{t}))^{\frac{1}{2}}} e^{-\frac{m_a d_t(x_0, y_0)^2}{16t}}$$

□

Remark 3.7. The constant $\frac{5}{8}$ and $\frac{1}{4}$ are far from being optimal. Moreover, assumption H1 does not seem to be necessary. Hypothesis H2 is a sufficient condition for the existence in short time of the Ricci flow [8].

Remark 3.8. The above estimate also produce a control for the heat kernel of the conjugate heat equation.

Remark 3.9. We could apply the same strategies for a general family of metric.

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